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Genus formula for generalized offset curves

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Abstract

In this paper, we present a formula for computing the genus of irreducible generalized offset curves to projective irreducible plane curves with only affine ordinary singularities over an algebraically closed field. The formula expresses the genus of the offset by means of the degree and the genus of the original curve.

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1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. We consider $\mathbb{K}^2$ as the metric affine space induced by the inner product $B((x_1,x_2),(y_1,y_2))=x_1y_1+x_2y_2$. In this context, the circle of center $(a_1,a_2)\in\mathbb{K}^2$ and radius $d\in\mathbb{K}$ is the plane curve defined by $(x_1-a_1)^2+(x_2-a_2)^2=d^2$. If we embed $\mathbb{K}^2$ in the projective plane $\mathbb{P}^2(\mathbb{K})$, a circle can be viewed as a conic passing through the points at the infinity of projective coordinates $(x_0:x_1:x_2)=(0:\pm\sqrt{-1}:1)$, called *cyclic points*. We will say that the distance between the points $\bar{x},\bar{y}\in\mathbb{K}^2$ is $d\in\mathbb{K}$ if $\bar{y}$ is on the circle of center $\bar{x}$ and radius d (notice that the distance is hence defined up to the sign). Finally, a linear isometry is a linear transformation preserving the inner product B. Clearly, an isometry can be identify with an orthogonal matrix, hence of determinant ±1. Thus, when the determinant is $+1$ we say that the isometry is direct. For further details, on metric affine spaces, see [10].

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Let us fix a direct isometry A in $\mathbb{K}^2$, and an irreducible affine plane curve $\mathscr{C}$ over $\mathbb{K}$, different of the lines $y_1 \pm \sqrt{-1}y_2+c$, and defined by the polynomial $f \in \mathbb{K}[y_1, y_2]$. Then, we consider the set $\mathscr{A}_d^A(\mathscr{C})$ in $\mathbb{K}^2$ given by the intersection points of the circles of radius d centered at each point $P \in \mathscr{C}$ such that the normal line to $\mathscr{C}$ at P exists, and the line passing through P in the direction determined by the vector $\mathscr{N}(P)\cdot A$, where $\mathscr{N}(P)$ is a normal vector to $\mathscr{C}$ at P. $\mathscr{A}_d^A(\mathscr{C})$ is clearly constructible since it is the natural π_1-projection of the constructible set

$$\mathscr{B}_d^A(\mathscr{C}) = \left\{ (x_1,x_2),(y_1,y_2) \in \mathbb{K}^2 \times \mathbb{K}^2 \;\middle|\; \begin{array}{l} f(y_1,y_2)=0 \\ (x_1-y_1)^2+(x_2-y_2)^2=d^2 \\ (x_1-y_1)M_2(y_1,y_2) \\ \quad =(x_2-y_2)M_1(y_1,y_2) \end{array} \right\},$$

where $(M_1,M_2)=\mathscr{N}(y_1,y_2)\cdot A$. In this situation, the *generalized offset curve to* $\mathscr{C}$ at distance d in the direction determined by A is defined as the Zariski closure of $\mathscr{A}_d^A(\mathscr{C})$ in $\mathbb{K}^2$, and we denote it by $\mathscr{O}_d^A(\mathscr{C})$. We will work mainly with its projective closure $\overline{\mathscr{O}_d^A(\mathscr{C})}$, which can be obtained equivalently as the π_1-projection of the projective closure of $\mathscr{B}_d^A(\mathscr{C})$. Note that if there exist infinitely many points P and $\mathscr{C}$ such that the normal vectors to $\mathscr{C}$ at P are isotropic, then the previous construction yields the empty set. However, it is relatively easy to prove that the only irreducible plane curves with this property are the lines $y_1 \pm \sqrt{-1}y_2+c$, i.e. those passing through the cyclic points. It is also easy to see that the only case in which the offset contains a zero-dimensional component is when the original curve is a circle, the isometry is the identity, and the distance is the radius of the circle. We will assume this not to be the case and we will always consider that $d \neq 0$.

If $\mathbb{K}=\mathbb{C}$, $\mathscr{C}$ is a real curve and A is the identity matrix then the previous definition yields the classical notion of offset curves at distance d (see [5]), that essentially corresponds to the envelope of the system of circles centered at the points of $\mathscr{C}$ with fixed radius d. Note that in this case the metric we are using for $\mathbb{C}^2$ is not the hermitic standard one.

From the definition it follows that $\mathscr{O}_d^A(\mathscr{C})$ has at most two components (see [2, 3]). Furthermore, if $\mathscr{C}$ is given by a rational parametrization $\mathscr{P}(t)$, then $\mathscr{A}_d^A(\mathscr{C})$ is essentially the set in $\mathbb{K}^2$ generated by the formula $\mathscr{P}(t)\pm d\mathscr{N}(t)\cdot A/\|\mathscr{N}(t)\|$, where $\mathscr{N}(t)$ is the normal vector to $\mathscr{C}$ associated with the parametrization $\mathscr{P}(t)$. In this formula, by abuse of notation, for every non-isotropic $\bar{x} \in \mathbb{K}^2$ we write $\|\bar{x}\|$ to express any of the two numbers such that $\|\bar{x}\|^2=B(\bar{x},\bar{x})$. If $\bar{x} \in \mathbb{K}^2$ is isotropic, then we write $\|\bar{x}\|=0$. Similarly, one may introduce the notion of offset to hypersurfaces (see [1, 8]).

Some algebraic properties on offsets to plane curves, formerly called parallel curves, were already known to classical geometers [9]. Currently, the study of algebraic and geometric properties on offsets is still an active research area. The reason is that they arise in practical applications as tolerance analysis, geometric control, robot path-planning and numerical-control machining problems; like the description of the curve that a cylindrical tool executes when it moves through a prescribed path. In [2, 3] the basic geometric and topological properties are analyzed, and algorithmic approaches to compute the implicit equation are provided. Similarly, in [5] the computation of implicit equations for offsets to surfaces is also analyzed.

From the point of view of applications, offsets to rational varieties are specially interesting. The main problem in this context is to guarantee the rationality and to construct direct parametrization algorithms for offset varieties; i.e. algorithms that parametrize rational offsets by means of a parametrization of the original rational curve. The reason is that, in order to guarantee the compatibility of data structures and algorithms, rational parametrizations of inputs (the original curves) and outputs (the offset curves) are required. The main difficulty is that the rationality of the original variety is not preserved (in general) when the offset is considered. For instance, offsets to parabolas are rational, offsets to circles decompose in two rational components, while offsets to ellipses and hyperbolas are elliptic curves. Recently, several authors have addressed these problems. In [7] explicit formulas to produce rational plane and rational surfaces whose offsets are rational are given. In [6] a characterization of polynomial offset plane curves and rational plane curves is given, and explicit representations are presented. In [1] a complete characterization of the unirationality of the components of the generalized offset to a hypersurface is presented, and direct parametrization algorithms are derived.

Therefore, in particular, this phenomenon implies that the (geometric) genus of the original curve is not preserved when offseting. Thus, the natural question of analyzing the relation between the genus of the original curve and its offset arises. If $\mathcal{O}_d^A(\mathscr{C})$ is reducible, the genus of its components can be essentially studied from the results in [1]. More precisely, if $\mathscr{C}$ is rational and the offset is reducible, then each component is rational. Furthermore, if $\mathscr{C}$ has genus g and its offset is reducible, then the simple components of $\mathcal{O}_d^A(\mathscr{C})$ (an irreducible component $\mathscr{M}$ of $\mathcal{O}_d^A(\mathscr{C})$ is called *simple* if almost every point $Q \in \mathscr{M}$ is generated by exactly one point $P \in \mathscr{C}$, otherwise $\mathscr{M}$ is called *special*, see [1]) are birationally equivalent to $\mathscr{C}$, and therefore of genus g. However, we do not know the behaviour of the genus of the special components if g is positive.

In this paper, we study the genus of irreducible offsets, and we present a formula for computing the genus of the generalized offset curve by means of the degree and the genus of the original curve, under some generality conditions. These generality conditions of the original curve (see Lemma 1) are clearly needed since the genus of the offset does not depend only on the degree and the genus of the original curve. For example the genus of the offset of a smooth conic is different for a parabola or a hyperbola; this shows why condition (2) in Lemma 1 is needed. It seems reasonable that if some of our conditions are removed the same kind of techniques would allow to derive the genus of the offset curve at each particular case.

2. Genus formula

In this section, the genus formula is proved. For this purpose, let $\bar{\mathscr{C}}$ be an irreducible projective algebraic plane curve over $\mathbb{K}$ ($\mathscr{C}$ will denote the affine curve corresponding to $\bar{\mathscr{C}}$). We start with the following lemma that introduces the general conditions on the curve $\bar{\mathscr{C}}$ that we need to derive the genus formula.

Lemma 1. *Let $\bar{\mathscr{C}}$ be of degree n, defined by the homogeneous polynomial $F(y_0, y_1, y_2) \in \mathbb{K}[y_0, y_1, y_2]$, satisfying that*

(1) All the singularities of $\bar{\mathscr{C}}$ are affine and ordinary.

(2) The line $y_0 = 0$ is not tangent to $\bar{\mathscr{C}}$.

(3) The curve $\bar{\mathscr{C}}$, the tangent lines to $\bar{\mathscr{C}}$ at the flex points (i.e. the flex lines), and the tangent lines to $\bar{\mathscr{C}}$ at the singularities do not pass through the cyclic points.

Then, if $\bar{\mathscr{C}}'$ is the curve defined by $(\partial F/\partial y_1)^2 + (\partial F/\partial y_2)^2$, it holds that degree of $\bar{\mathscr{C}}'$ is exactly $2(n-1)$, that all the intersections of $\bar{\mathscr{C}}$ and $\bar{\mathscr{C}}'$ are affine, and that the multiplicity of intersection at each intersection point is minimal (i.e. it is $2r(r-1)$ at an r-fold point of $\bar{\mathscr{C}}$ if $r \geq 2$, or it is one if $r = 1$).

Proof. The sketch of the proof is as follows: first we show that the degree of $\bar{\mathscr{C}}'$ is $2(n-1)$, then we see that all the intersections points are affine, and finally we prove that the multiplicity of intersection at each intersection point is minimal. For this purpose, we introduce the following notation: $G(y_0, y_1, y_2) = (\partial F/\partial y_1)^2 + (\partial F/\partial y_2)^2$, $G_1(y_0, y_1, y_2) = \partial F/\partial y_1 + \sqrt{-1}(\partial F/\partial y_2)$, $G_2(y_0, y_1, y_2) = \partial F/\partial y_1 - \sqrt{-1}(\partial F/\partial y_2)$, and $f(y_1, y_2) = F(1, y_1, y_2)$, $g(y_1, y_2) = G(1, y_1, y_2)$, $g_1(y_1, y_2) = G_1(1, y_1, y_2)$, $g_2(y_1, y_2) = G_2(1, y_1, y_2)$.

Let us see that the degree of $\bar{\mathscr{C}}'$ is $2(n-1)$. Indeed, let $f(y_1, y_2) = f_n(y_1, y_2) + \cdots + f_0(y_1, y_2)$, where $f_k = \sum_{j=0}^{k} a_{k,j} y_1^j y_2^{k-j} \in \mathbb{K}[y_1, y_2]$ is the homogeneous component of degree k of the polynomial f. Let us suppose that $deg(G) < 2(n-1)$. Then $deg(g) < 2(n-1)$, and hence $(\partial f_n^2/\partial y_1) + (\partial f_n^2/\partial y_2) = 0$. Thus, either $(\partial f_n/\partial y_1) + \sqrt{-1}(\partial f_n/\partial y_2) = 0$ or $(\partial f_n/\partial y_1) - \sqrt{-1}(\partial f_n/\partial y_2) = 0$. Let us assume $(\partial f_n/\partial y_1) + \sqrt{-1}(\partial f_n/\partial y_2) = 0$ (similarly, if $(\partial f_n/\partial y_1) - \sqrt{-1}(\partial f_n/\partial y_2) = 0$). This implies, after some computations, that

$$a_{n,n-j} = -\frac{\sqrt{-1}(j+1)a_{n,n-j-1}}{n-j} \quad \text{for } j = 0, \ldots, n-1.$$

Therefore, one deduces that

$$a_{n,s} = (-\sqrt{-1})^s \binom{n}{s} a_{n,0} \quad \text{for } s = 1, \ldots, n.$$

Thus, $f_n(y_1, y_2) = (y_2 - \sqrt{-1}y_1)^n a_{n,0}$, which is impossible since $\bar{\mathscr{C}}$ does not pass through the cyclic points. Hence, $deg(G) = 2(n-1)$.

We now prove that all the intersection points of $\bar{\mathscr{C}}$ and $\bar{\mathscr{C}}'$ are affine. Let $P = (0 : a : 1)$ be an intersection point of $\bar{\mathscr{C}}$ and $\bar{\mathscr{C}}'$. Then, $F(P) = 0$, and either $G_1(P) = 0$ or $G_2(P) = 0$ (note that G_1 and G_2 can not vanish simultaneously at P, since it would imply that $y_0 = 0$ is tangent to $\bar{\mathscr{C}}$ at P, which is impossible by hypothesis). Let us suppose that $G_1(P) = 0$ (similarly if $G_2(P) = 0$). Thus one has that $res_{y_1}(f_n, h_{n-1}) = 0$, where $f_n(y_1, y_2) = F(0, y_1, y_2)$, and $h_{n-1}(y_1, y_2) = G_1(0, y_1, y_2)$, $(res_{y_1}(f_n, h_{n-1})$

denotes the resultant of f_n and h_{n-1} with respect to y_1). On the other hand, applying Euler's formula, one obtains that $nf_n = y_1(\partial f_n/\partial y_1) + y_2(\partial f_n/\partial y_2)$. This implies that

$$h_{n-1} = \frac{\partial f_n}{\partial y_1} + \sqrt{-1}\frac{\partial f_n}{\partial y_2} = \left(1 - \sqrt{-1}\frac{y_1}{y_2}\right)\frac{\partial f_n}{\partial y_1} + \sqrt{-1}\frac{n}{y_2}f_n.$$

Thus,

$$\begin{aligned} res_{y_1}(f_n, h_{n-1}) &= res_{y_1}\left(f_n, \left(1 - \sqrt{-1}\frac{y_1}{y_2}\right)\frac{\partial f_n}{\partial y_1}\right) \\ &= res_{y_1}\left(f_n, \left(1 - \sqrt{-1}\frac{y_1}{y_2}\right)\right) \cdot res_{y_1}\left(f_n, \frac{\partial f_n}{\partial y_1}\right) = 0. \end{aligned}$$

Therefore, either $res_{y_1}(f_n, (1 - \sqrt{-1}(y_1/y_2))) = 0$ or $res_{y_1}(f_n, \partial f_n/\partial y_1) = 0$. However, if $res_{y_1}(f_n, (1 - \sqrt{-1}(y_1/y_2))) = 0$ one has that $f_n(y_2/\sqrt{-1}, y_2) = y_2^n f_n(1 - \sqrt{-1}, 1) = 0$; but this implies that $\bar{\mathscr{C}}$ passes through the cyclic point $(0 : -\sqrt{-1} : 1)$, which is impossible by hypothesis. On the other hand, if $res_{y_1}(f_n, \partial f_n/\partial y_1) = 0$, taking into account that $\bar{\mathscr{C}}$ does not have singularities at infinity, one deduces that the line $y_0 = 0$ is tangent to $\bar{\mathscr{C}}$ at P, which is also impossible by hypothesis. Thus, $\bar{\mathscr{C}}$ and $\bar{\mathscr{C}}'$ have no intersection at infinity.

We will see now that the multiplicity of intersection at each intersection point is minimal. Let $P = (1 : a : b)$ be an intersection point of $\bar{\mathscr{C}}$ and $\bar{\mathscr{C}}'$. We distinguish two cases: P is a simple point of $\bar{\mathscr{C}}$, or P is a singularity of $\bar{\mathscr{C}}$. Let $P \in \bar{\mathscr{C}} \cap \bar{\mathscr{C}}'$ be simple on $\bar{\mathscr{C}}$, then $f(a,b) = 0$ and either $g_1(a,b) = 0$ or $g_2(a,b) = 0$ (note that g_1 and g_2 cannot vanish simultaneously at (a,b), since it would imply that P is a singularity of $\bar{\mathscr{C}}$). Let us suppose that $g_1(a,b) = 0$ (similarly, if $g_2(a,b) = 0$). We write $f(x,y)$ as its Taylor expansion up to degree two:

$$\begin{aligned} f(x,y) = {} & b_1(y_1 - a) + b_2(y_2 - b) + c_1(y_1 - a)^2 + c_2(y_1 - a)(y_2 - b) \\ & + c_3(y_2 - b)^2 + \cdots. \end{aligned}$$

Then

$$\begin{aligned} g_1(y_1, y_2) = {} & (b_1 + \sqrt{-1}b_2) + (2c_1 + \sqrt{-1}c_2)(y_1 - a) \\ & + (c_2 + 2\sqrt{-1}c_3)(y_2 - b) + \cdots \end{aligned}$$

and therefore, $g_1(a,b) = b_1 + \sqrt{-1}b_2 = 0$. Thus, the tangent line to $\bar{\mathscr{C}}$ at P passes through the cyclic point $(0 : -\sqrt{-1} : 1)$. Therefore, taking into account that no flex line passes through the cyclic points, one has that P is not a flex point. We now observe that P is also a simple point of the projective curve defined by G_1. Indeed: if P is not simple of $\bar{\mathscr{C}}'$ then $2c_1 + \sqrt{-1}c_2 = 0$, $c_2 + 2\sqrt{-1}c_3 = 0$. Hence, $c_1 + \sqrt{-1}c_2 - c_3 = 0$, and therefore P is a flex point, which is impossible.

In this situation, it only remains to prove that the tangent to $\bar{\mathscr{C}}$ at P and the tangent to $\bar{\mathscr{C}}'$ at P are not parallel. But, using that $b_1+\sqrt{-1}b_2=0$, one has that

$$\det\begin{pmatrix} b_1 & b_2 \\ 2c_1+\sqrt{-1}c_2 & c_2+2\sqrt{-1}c_3 \end{pmatrix} = -2b_2(c_1+\sqrt{-1}c_2-c_3)$$

which is not zero, since $b_2\neq 0$ (note that if $b_2=0$ then $b_1=0$, and hence P would be singular on $\bar{\mathscr{C}}$), and $c_1+\sqrt{-1}c_2-c_3\neq 0$ (note that P is not a flex point of $\bar{\mathscr{C}}$). Therefore, the tangents are not parallel. Summarizing, one concludes that the intersection multiplicity at each the intersection simple point, is one.

Finally, we prove that the multiplicity of intersection at the singularities of $\bar{\mathscr{C}}$ is also minimal. Let $P=(1:a:b)$ be an r-fold point of $\bar{\mathscr{C}}$. Then, it is clear that P is a point of multiplicity at least $(r-1)$ on both curves G_1 and G_2. Let us see that, in fact, P is an $(r-1)$-fold point of G_1 (similarly for G_2). If one assumes that P has multiplicity r on G_1, then

$$\frac{\partial g}{\partial y_1^k\partial y_2^s}(a,b)=\frac{\partial f}{\partial y_1^{k+1}\partial y^s}(a,b)+\sqrt{-1}\frac{\partial f}{\partial x^k\partial y^{s+1}}(a,b)=0 \quad \text{for } k+s=r-1.$$

This implies that, after some computations,

$$\frac{\partial f}{\partial y_1^k\partial y_2^{r-k}}(a,b)=(-\sqrt{-1})^k\frac{\partial f}{\partial y_2^r}(a,b) \quad \text{for } k=1,\ldots,r.$$

On the other hand, taking into account that the tangents to $\bar{\mathscr{C}}$ at P are

$$T(y_0,y_1,y_2)=\sum_{s=0}^{r}\binom{r}{s}\frac{\partial f}{\partial y_1^s\partial y_2^{r-s}}(a,b)(y_1-ay_0)^s(y_2-by_0)^{r-s},$$

one has that

$$T(0,-\sqrt{-1},1)=\sum_{s=0}^{r}\binom{r}{s}\frac{\partial f}{\partial y_1^s\partial y_2^{r-s}}(a,b)(1-\sqrt{-1})^s=(-1+1)^r\frac{\partial f}{\partial y_2^r}(a,b)=0.$$

Therefore, at least one tangent to $\bar{\mathscr{C}}$ at P passes through one cyclic point, which is impossible by hypothesis.

Now, we prove that the multiplicity of intersection at P is minimal. For this purpose, we check that the tangents to $\bar{\mathscr{C}}$ at P and to G_1 at P are not parallel (similarly for G_2). Let the polynomial T (that defines the tangents to $\bar{\mathscr{C}}$ at P) factor as

$$T(y_0,y_1,y_2)=\prod_{j=1}^{r}(\alpha_j y_1+\beta_j y_2+\gamma_j y_0).$$

Then, the tangents $T'(y_0, y_1, y_2)$ to G_1 at P are

$$\begin{aligned} T' &= \sum_{k=1}^{r} \alpha_k \prod_{j \neq k} (\alpha_j y_1 + \beta_j y_2 + \gamma_j y_0) + \sqrt{-1} \sum_{k=1}^{r} \beta_k \prod_{j \neq k} (\alpha_j y_1 + \beta_j y_2 + \gamma_j y_0) \\ &= \sum_{k=1}^{r} (\alpha_k + \sqrt{-1}\beta_k) \prod_{j \neq k} (\alpha_j y_1 + \beta_j y_2 + \gamma_j y_0). \end{aligned}$$

In this situation, let us assume that there exists $s \in \{1, \ldots, r\}$ such that $(\alpha_s y_1 + \beta_s y_2 + \gamma_s y_0)$ is parallel to one factor of $T'(y_0, y_1, y_2)$. If $\alpha_s \neq 0$ (similarly if $\alpha_s = 0$), one has that

$$\begin{aligned} T'\left(1, -\frac{\gamma_s}{\alpha_s} - \frac{\beta_s}{\alpha_s} y_2, y_2\right) &= \sum_{k=1}^{r} (\alpha_k + \sqrt{-1}\beta_k) \prod_{j \neq k} \left(-\frac{\alpha_j \gamma_s}{\alpha_s} - \frac{\alpha_j \beta_s}{\alpha_s} y_2 + \beta_j y_2 + \gamma_j\right) \\ &= (\alpha_s + \sqrt{-1}\beta_s) \prod_{j \neq s} \left(-\frac{\alpha_j \gamma_s}{\alpha_s} - \frac{\alpha_j \beta_s}{\alpha_s} y_2 + \beta_j y_2 + \gamma_j\right) = 0 \end{aligned}$$

which is impossible, since $(\alpha_s + \sqrt{-1}\beta_s) \neq 0$ (note that tangents to $\bar{\mathscr{C}}$ at P passes through the cyclic points), and $\prod_{j \neq s}(-(\alpha_j \gamma_s/\alpha_s) - (\alpha_j \beta_s/\alpha_s) y_2 + \beta_j y_2 + \gamma_j) \neq 0$ (note that P is ordinary). Therefore, the multiplicity of intersection of $\bar{\mathscr{C}}$ and $\bar{\mathscr{C}}'$ at P is $2r(r-1)$. $\square$

Now, we proceed to study the genus of generalized offset curves. For this purpose, in the sequel, let $\bar{\mathscr{C}}$ be an irreducible projective plane curve satisfying that hypotheses of Lemma 1, let A be a matrix defining a direct isometry in $\mathbb{K}^2$, and let $\overline{\mathcal{O}_d^A(\mathscr{C})}$ be the projective closure of the offset curve to $\mathscr{C}$ (note that $\mathscr{C}$ is different of the lines $y_1 \pm \sqrt{-1}\, y_2 + c$, since it does not pass through the cyclic points). Then, we consider the constructible set $\mathscr{B}_d^A(\mathscr{C})$, introduced in Section 1, its projective closure $\overline{\mathscr{B}_d^A(\mathscr{C})}$, and the following diagram:

$$\begin{tikzcd} & \overline{\mathscr{B}_d^A(\mathscr{C})} \subset \mathbb{P}(\mathbb{K})^2 \times \mathbb{P}(\mathbb{K})^2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \overline{\mathcal{O}_d^A(\mathscr{C})} \subset \mathbb{P}(\mathbb{K})^2 & & \bar{\mathscr{C}} \subset \mathbb{P}(\mathbb{K})^2 \end{tikzcd} \qquad \text{(I)}$$

where π_1 and π_2 are the natural projections. We also consider the induced map $\tilde{\pi}_2 : \tilde{\mathscr{B}}_d^A(\mathscr{C}) \to \tilde{\mathscr{C}}$ between the desingularization curves of $\overline{\mathscr{B}_d^A(\mathscr{C})}$ and $\bar{\mathscr{C}}$. We will view the points of a desingularization curve as the set of places of the given curve. A place centered at a point P on a curve, following the definition given in [11], is an equivalence class of local parametrizations by formal power series of $\mathscr{C}$ at P. The idea now is to derive the genus of $\tilde{\mathscr{B}}_d^A(\mathscr{C})$ (which is birational to the offset curve when this is simple) from the genus of $\tilde{\mathscr{C}}$ (which is the same as the one of $\bar{\mathscr{C}}$) by applying the Riemann-Hurwitz theorem to the map $\tilde{\pi}_2$. For this purpose, we need to study the branch locus of $\tilde{\pi}_2$, which we do in the next lemma.

Lemma 2. *Let π_2 be the projection in diagram* (I), *let $\tilde{\pi}_2$ be its desingularization, and let $F \in \mathbb{K}[y_0, y_1, y_2]$ be the homogeneous polynomial defining $\bar{\mathscr{C}}$. Then, in the hypotheses of Lemma* 1, *it holds that*

(1) *a place at an affine point P of $\bar{\mathscr{C}}$ (i.e. a point of $\mathscr{C}$) is a branch point for $\tilde{\pi}_2$ if and only if P is simple on $\bar{\mathscr{C}}$ and $(\partial F/\partial y_1)(P)^2 + (\partial F/\partial y_2)(P)^2 = 0$;*

(2) *every point P of $\bar{\mathscr{C}}$ at infinity is a branch point for π_2. Furthermore, $\pi_2^{-1}(P)$ is an ordinary double point of $\overline{\mathscr{B}_d^A(\mathscr{C})}$, hence it is not a branch point for $\tilde{\pi}_2$.*

Proof. We have to find those places $\mathscr{P}$ at points P of $\bar{\mathscr{C}}$ such that $\tilde{\pi}_2^{-1}(\mathscr{P})$ contains exactly one point on $\tilde{\mathscr{B}}_d^A(\mathscr{C})$.

(1) Let $P = (1 : a_0 : b_0) \in \bar{\mathscr{C}}$ be an r-fold point, $r \geq 1$, and let

$$\mathscr{P}(t) = \begin{cases} y_0 = 1, \\ y_1 = g_1(t) = \sum_{i\geq 0} a_i t^i, \\ y_2 = g_2(t) = \sum_{i \geq 0} b_i t^i, \end{cases}$$

be one of the places of $\bar{\mathscr{C}}$ with center P. Note that $\bar{\mathscr{C}}$ only has ordinary singularities. Thus, any place centered at a singular point of $\bar{\mathscr{C}}$ is simple, and therefore a_1 and b_1 can not vanish simultaneously. Then, taking into account the definition of generalized offset, one has that

$$\mathscr{Q}(t) = \begin{cases} x_0 = \pm\sqrt{g_1'^2 + g_2'^2}, \\ x_1 = \pm g_1\sqrt{g_1'^2 + g_2'^2} - d(a_{11}g_2' - a_{21}g_1'), \\ x_2 = \pm g_2\sqrt{g_1'^2 + g_2'^2} - d(a_{12}g_2' - a_{22}g_1'), \\ y_0 = 1, \\ y_1 = g_1(t), \\ y_2 = g_2(t), \end{cases}$$

parametrizes locally $\overline{\mathscr{B}_d^A(\mathscr{C})}$, where $A = (a_{ij})$. Therefore,

$$\pi_2^{-1}(P) = \begin{cases} x_0 = \pm\sqrt{a_1^2 + b_1^2}, \\ x_1 = \pm a_0\sqrt{a_1^2 + b_1^2} - d(a_{11}b_1 - a_{21}a_1), \\ x_2 = \pm b_0\sqrt{a_1^2 + b_1^2} - d(a_{12}b_1 - a_{22}a_1), \\ y_0 = 1, \\ y_1 = a_0, \\ y_2 = b_0. \end{cases}$$

Now, we observe that $(a_{11}b_1 - a_{21}a_1)$ and $(a_{12}b_1 - a_{22}a_1)$ can not vanish simultaneously (note that, since A is an isometry it would imply that $(-g_2', g_1') = \mathbf{0}$, and therefore $a_1 = b_1 = 0$, which is impossible). Therefore, if $a_1^2 + b_1^2 \neq 0$ then $\pi_2^{-1}(P)$ contains two different points; and hence $\mathscr{P}$ is not a branch point. Also notice that this is the case when P is a multiple point, since we are assuming that no tangent line of $\bar{\mathscr{C}}$ at P passes through a cyclic point.

Now if $(\partial F/\partial y_1)(P)^2 + (\partial F/\partial y_2)(P)^2 = a_1^2 + b_1^2 = 0$, in order to prove that $\mathscr{P}$ is a branch point of $\tilde{\pi}_2$, we need to show that the two local parametrizations that $\mathscr{Q}(t)$ defines correspond to the same place. For this purpose, we observe that, since $a_1^2 + b_1^2 = 0$, then $\mathscr{Q}(t)$ is not a formal power series. However, making a change of parameter in $\mathscr{Q}(t)$, we show that both parametrizations generate the same branch. More precisely, since $\bar{\mathscr{C}}$ satisfies the hypotheses of Lemma 1, and taking into account that P is an intersection point of $\bar{\mathscr{C}}$ and the curve $\bar{\mathscr{C}}'$ defined by $(\partial F/\partial y_1)^2 + (\partial F/\partial y_2)^2$, one has that the multiplicity of intersection of $\bar{\mathscr{C}}$ and $\bar{\mathscr{C}}'$ at the simple place $\mathscr{P}(t)$ is one. Therefore, if $f(y_1, y_2) = F(1, y_1, y_2)$, it holds that $(\partial f/\partial y_1)(g_1, g_2)^2 + (\partial f/\partial y_2)(g_1, g_2)^2$ is a formal power series of order one. On the other hand, it is easy to check that

$$\frac{\partial f}{\partial y_1}(g_1, g_2)^2 + \frac{\partial f}{\partial y_2}(g_1, g_2)^2 = \frac{\partial f}{\partial y_2}(g_1, g_2)^2 (g_1'^2 + g_2'^2).$$

Thus, since $g_1'^2(0) + g_2'^2(0) = 0$, one deduces that the order of $g_1'^2 + g_2'^2$ is one. In this situation, $\sqrt{g_1'^2 + g_2'^2}$ can be written as $\sqrt{g_1'^2 + g_2'^2} = \sqrt{t} \cdot h(t)$, where $h(t)$ is a unit. Thus, $\mathscr{Q}(t)$ can be parametrized as

$$\mathscr{Q}(s^2) = \begin{cases} x_0 = \pm s \cdot h(s^2), \\ x_1 = \pm s \cdot g_1(s^2) h(s^2) - d(a_{11} g_2'(s^2) - a_{21} g_1'(s^2)), \\ x_2 = \pm s \cdot g_2(s^2) h(s^2) - d(a_{12} g_2'(s^2) - a_{22} g_1'(s^2)), \\ y_0 = 1, \\ y_1 = g_1(s^2), \\ y_2 = g_2(s^2). \end{cases}$$

Now, it is clear that both local parametrizations define the same place.

(2) Let $P = (0 : a_0 : 1)$ a point of $\bar{\mathscr{C}}$ at infinity (note that $\bar{\mathscr{C}}$ has no singularities at infinity) and let

$$\mathscr{P}(t) = \begin{cases} y_0 = \tilde{g}_0(t) = \sum_{i \geq 1} c_i t^i, \\ y_1 = \tilde{g}_1(t), \\ y_2 = 1 \end{cases}$$

be the place with center P. We observe that $c_1 \neq 0$, since $y_0 = 0$ is not tangent to $\bar{\mathscr{C}}$. Thus, $\mathscr{P}(t)$ can be written as

$$\mathscr{P}(t) = \begin{cases} y_0 = 1, \\ y_1 = \frac{g_1(t)}{t}, \\ y_2 = \frac{g_2(t)}{t}, \end{cases}$$

where $g_2(t)$ is a unit. Then,

$$\mathscr{Q}(t) = \begin{cases} x_0 = 1, \\ x_1 = \frac{g_1(t)}{t} - d\frac{(\Delta_2 a_{11} - \Delta_1 a_{21})}{\sqrt{\Delta_1^2 + \Delta_2^2}}, \\ x_2 = \frac{g_2(t)}{t} - d\frac{(\Delta_2 a_{12} - \Delta_1 a_{22})}{\sqrt{\Delta_1^2 + \Delta_2^2}}, \\ y_0 = 1, \\ y_1 = \frac{g_1(t)}{t}, \\ y_2 = \frac{g_2(t)}{t}, \end{cases}$$

where $\Delta_1(t) = g_1' t - g_1$ and $\Delta_2(t) = g_2' t - g_2$. Now, taking into account that $(t, g_1(t), g_2(t))$ parametrizes $\bar{\mathscr{C}}$, and that $\bar{\mathscr{C}}$ does not pass through the cyclic points, one deduces that

$$\Delta_1^2(0) + \Delta_2^2(0) = g_1^2(0) + g_2^2(0) \neq 0.$$

Therefore, $\pi_2^{-1}(P) = \{(0 : g_1(0) : g_2(0))\} \times \{P\} = \{(P,P)\}$, and hence P is a branch point for π_2.

Finally, let us see that (P,P) is an ordinary double point of $\overline{\mathscr{B}_d^A(\mathscr{C})}$. For this purpose, one only has to observe that both parametrizations defined by $\mathscr{Q}(t)$ are non-equivalent formal power series (note that $\Delta_1(0)^2 + \Delta_2(0)^2 \neq 0$) defining two different places. □

The next theorem gives a formula for computing the genus of the generalized offset curve to a curve satisfying the hypotheses of Lemma 1. In order to have that $\bar{\mathscr{B}}_d^A(\mathscr{C})$ is birationally equivalent to the offset curve we need to impose that the offset is simple. In the classical case, it is relatively easy to prove that this condition is implied by the irreducibility of the offset. We conjecture that this is also the case for generalized offsets.

Theorem 1. *Let $\bar{\mathscr{C}}$ be a curve of degree n satisfying the hypotheses of Lemma 1 and such that $\mathcal{O}_d^A(\mathscr{C})$ is irreducible and simple. Let $r_1, \ldots, r_s$ be the multiplicities of its singular points. Then the generalized offset at distance d has genus $g(\mathcal{O}_d^A(\mathscr{C})) = 2n^2 - 4n + 1 - 2\sum_{j=1}^{s} r_j(r_j - 1)$.*

Proof. We just apply the Riemann-Hurwitz theorem (see e.g. [4]) to the map $\tilde{\pi}_2 : \tilde{\mathscr{B}}_d^A(\mathscr{C}) \to \tilde{\mathscr{C}}$ to obtain that $2g(\tilde{\mathscr{B}}_d^A(\mathscr{C})) - 2 = 2(2g(\tilde{\mathscr{C}}) - 2) + deg(\text{branch locus})$. The theorem follows now at once from this equality and the following three remarks:

(1) Since $\mathcal{O}_d^A(\mathscr{C})$ is simple and irreducible, the map π_1 in diagram (I) is birational, so that its genus is $g(\mathcal{O}_d^A(\mathscr{C})) = g(\overline{\mathscr{B}_d^A(\mathscr{C})}) = g(\tilde{\mathscr{B}}_d^A(\mathscr{C}))$.

(2) Since $\bar{\mathscr{C}}$ is a plane curve of degree n with only ordinary singularities of multiplicities $r_1, \ldots, r_s$, it follows that its genus is $g(\bar{\mathscr{C}}) = g(\tilde{\mathscr{C}}) = \frac{1}{2}(n-1)(n-2) - \sum_{j=1}^{s} \frac{1}{2} r_j(r_j - 1)$.

(3) According to Lemma 2, the branch points of $\tilde{\pi}_2$ are the smooth points of $\bar{\mathscr{C}}$ that are also in $\bar{\mathscr{C}}'$. The number of points in $\bar{\mathscr{C}} \cap \bar{\mathscr{C}}'$, counted with multiplicities, is $2n(n-1)$ (and they all are affine by Lemma 1). If one of these points has multiplicity $r_j \geq 2$ as a point of $\bar{\mathscr{C}}$, then Lemma 1 implies that the r_j places of $\bar{\mathscr{C}}$ it produces (none of them a branch point) give a total contribution of multiplicity $2r_j(r_j - 1)$. Also from Lemma 1, at a smooth point of $\bar{\mathscr{C}}$ that is a branch point for $\tilde{\pi}_2$, the intersection multiplicity of $\bar{\mathscr{C}}$ and $\bar{\mathscr{C}}'$ is one. Summing all this up, we get that the number of branch points of $\tilde{\pi}_2$ is $2n(n-1) - \sum_{j=1}^{s} 2r_j(r_j - 1)$. This is also the degree of the branch locus, since $\tilde{\pi}_2$ is a double cover, so that any branch point appears with multiplicity one in the branch locus. □

This result can be immediately applied to compute the genus of the ellipses and hyperbolas.

Corollary. *Generalized offsets to ellipses and hyperbolas are elliptic curves.*

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